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Some Asymptotic Boundary Behavior of Solutions of Non-Linear Parabolic Initial-Boundary Value Problems

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For parabolic initial boundary value problems various results such as $\lim_{t\downarrow 0}\{(\partial u/\partial x)(0,t)/(\partial u_\alpha/\partial x)(0,t)\} = 1$, where u satisfies $\partial u/\partial t = a(u)(\partial^2 u/\partial x^2)$, $0 < x < 1$, $0 < t \leqslant T$, $u(x, 0) = 0$, $u(0, t) = f_1(t)$, $0 < t \leqslant T$, $u(1, t) = f_2(t)$, $0 < t \leqslant T$, u_α satisfies $(\partial u_\alpha/\partial t) = \alpha(\partial^2 u_\alpha/\partial x^2)$, $0 < x < 1$, $0 < t \leqslant T$, $u_\alpha(x, 0) = 0$, $u_\alpha(0, t) = f_1(t)$, $0 < t \leqslant T$, $u_\alpha(1, t) = f_2(t)$, $0 < t \leqslant T$, and $\alpha = a(0)$, are demonstrated via the maximum principle and potential theoretic estimates.

1. INTRODUCTION

Let $u = u(x, t)$ denote the solution to the initial boundary value problem of the first kind

$$\begin{aligned} \frac{\partial u}{\partial t} &= a(u)\frac{\partial^2 u}{\partial x^2}, && 0 < x < 1, \quad 0 < t \leqslant T, \\ u(x, 0) &= 0, && 0 \leqslant x \leqslant 1, \\ u(0, t) &= f_1(t), && 0 < t \leqslant T, \\ u(1, t) &= f_2(t), && 0 < t \leqslant T, \end{aligned} \tag{1.1}$$

and let $v = v(x, t)$ denote the solution to the initial boundary value problem of the second kind

$$\begin{aligned} \frac{\partial v}{\partial t} &= a(v)\frac{\partial^2 v}{\partial x^2}, && 0 < x < 1, \quad 0 < t \leqslant T, \\ v(x, 0) &= 0, && 0 \leqslant x \leqslant 1, \\ \frac{\partial v}{\partial x}(0, t) &= g_1(t), && 0 < t \leqslant T, \\ \frac{\partial v}{\partial x}(1, t) &= g_2(t), && 0 < t \leqslant T, \end{aligned} \tag{1.2}$$

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where $a = a(\cdot)$ is a known positive smooth function defined on $(-\infty, \infty)$ and where $f_i, g_i, i = 1, 2$, are known smooth functions which are defined on $0 \leqslant t \leqslant T$ and such that $f_i(0) = g_i(0) = 0$, $i = 1, 2$. In addition, let $U = U(x, t)$ denote the bounded solution of

$$\begin{aligned} &\frac{\partial U}{\partial t} = a(U)\frac{\partial^2 U}{\partial x^2}, && 0 < x < \infty, \quad 0 < t \leqslant T, \\ &U(x, 0) = 0, && 0 \leqslant x < \infty, \\ &U(0, t) = f_1(t), && 0 < t \leqslant T, \end{aligned} \tag{1.3}$$

and let $V = V(x, t)$ denote the bounded solution of

$$\begin{aligned} &\frac{\partial V}{\partial t} = a(V)\frac{\partial^2 V}{\partial x^2}, && 0 < x < \infty, \quad 0 < t \leqslant T, \\ &V(x, 0) = 0, && 0 \leqslant x < \infty \\ &\frac{\partial V}{\partial x}(0, t) = g_1(t), && 0 < t \leqslant T. \end{aligned} \tag{1.4}$$

By the term *solution* we mean functions which possess continuous first derivatives with respect to x and t in the closure of the domain under consideration, which possess a continuous second derivative with respect to x in the considered domain, and which satisfy the given equation and specified initial and boundary conditions. We assume the solutions u, v, U, and V exist and are unique. See [1, 4] for such results.

Setting

$$\alpha = a(0), \tag{1.5}$$

let $u_\alpha = u_\alpha(x, t)$ denote the classical solution of the problem

$$\begin{aligned} &\frac{\partial u_\alpha}{\partial t} = \alpha\frac{\partial^2 u_\alpha}{\partial x^2}, && 0 < x < 1, \quad 0 < t \leqslant T, \\ &u_\alpha(x, 0) = 0, && 0 \leqslant x \leqslant 1, \\ &u_\alpha(0, t) = f_1(t), && 0 < t \leqslant T, \\ &u_\alpha(1, t) = f_2(t), && 0 < t \leqslant T, \end{aligned} \tag{1.6}$$

and let $v_\alpha = v_\alpha(x, t)$ denote the classical solution of the problem

$$\begin{aligned} &\frac{\partial v_\alpha}{\partial t} = \alpha\frac{\partial^2 v_\alpha}{\partial v^2}, && 0 < x < 1, \quad 0 < t \leqslant T, \\ &v_\alpha(x, 0) = 0, && 0 \leqslant x \leqslant 1, \\ &\frac{\partial v_\alpha}{\partial x}(0, t) = g_1(t), && 0 < t \leqslant T, \\ &\frac{\partial v_\alpha}{\partial x}(1, t) = g_2(t), && 0 < t \leqslant T. \end{aligned} \tag{1.7}$$

The existence and uniqueness of the classical solutions u_α and v_α can be found in [7].

Also, let $U_\alpha = U_\alpha(x, t)$ denote the bounded classical solution of

$$\begin{aligned} \frac{\partial U_\alpha}{\partial t} &= \alpha \frac{\partial^2 U_\alpha}{\partial x^2}, && 0 < x < \infty, \quad 0 < t \leqslant T, \\ U_\alpha(x, 0) &= 0, && 0 \leqslant x < \infty, \\ U_\alpha(0, t) &= f_1(t), && 0 \leqslant t \leqslant T, \end{aligned} \tag{1.8}$$

and let $V_\alpha = V_\alpha(x, t)$ denote the bounded classical solution of

$$\begin{aligned} \frac{\partial V_\alpha}{\partial t} &= \alpha \frac{\partial^2 V_\alpha}{\partial x^2}, && 0 < x < \infty, \quad 0 < t \leqslant T, \\ V_\alpha(x, 0) &= 0, && 0 \leqslant x < \infty, \\ \frac{\partial V_\alpha}{\partial x}(0, t) &= g_1(t), && 0 < t \leqslant T. \end{aligned} \tag{1.9}$$

Formulas for the solution of (1.6) and the solution of (1.7) can be found in [7].

The purpose of this paper is to show a variety of asymptotic results concerning $(\partial u/\partial x)(0, t)$, $v(0, t)$, $(\partial U/\partial x)(0, t)$ and $V(0, t)$ as t tends to zero. In other words, we shall show that under certain assumptions upon the data f_i, and g_i, $i = 1, 2$, it follows that

$$\begin{aligned} &\lim_{t \downarrow 0} \left\{ \frac{\partial u}{\partial x}(0, t) \Big/ \frac{\partial U}{\partial x}(0, t) \right\} \\ &\quad = \lim_{t \downarrow 0} \left\{ \frac{\partial u}{\partial x}(0, t) \Big/ \frac{\partial u_\alpha}{\partial x}(0, t) \right\} = \lim_{t \downarrow 0} \left\{ \frac{\partial u}{\partial x}(0, t) \Big/ \frac{\partial U_\alpha}{\partial x}(0, t) \right\} \\ &\quad = \lim_{t \downarrow 0} \left\{ \frac{\partial U}{\partial x}(0, t) \Big/ \frac{\partial U_\alpha}{\partial x}(0, t) \right\} = \lim_{t \downarrow 0} \left\{ \frac{\partial u_\alpha}{\partial x}(0, t) \Big/ \frac{\partial U_\alpha}{\partial x}(0, t) \right\} = 1, \end{aligned} \tag{1.10}$$

and that

$$\begin{aligned} \lim_{t \downarrow 0} \frac{v(0, t)}{V(0, t)} &= \lim_{t \downarrow 0} \frac{v(0, t)}{v_\alpha(0, t)} = \lim_{t \downarrow 0} \frac{v(0, t)}{V_\alpha(0, t)} \\ &= \lim_{t \downarrow 0} \frac{V(0, t)}{V_\alpha(0, t)} = \lim_{t \downarrow 0} \frac{v_\alpha(0, t)}{V_\alpha(0, t)} = 1. \end{aligned} \tag{1.11}$$

The proof of these results will involve a use of the maximum principle coupled with classical formulas for the heat equation. In the next section conditions are imposed upon the data f_i and g_i, $i = 1, 2$, which insure that $\partial^2 u_\alpha/\partial x^2$ and $\partial^2 v_\alpha/\partial x^2$ are positive for $0 < x < 1$, $0 < t \leqslant T$. These results are employed in the third section in order to derive some comparison theorems which involve the bounding of u and v above and below with solutions to heat equations with diffusivities respectively greater than α and less than α. The fourth section is devoted to the analysis of computing the limits (1.10) and (1.11) including the results involving the solutions of the quarter plane problems.

2. Positivity of the Second Spatial Derivative of the Solution of the Heat Equation

From [7] we see that

$$u_\alpha(x, t) = -\int_0^t \frac{\partial M(x, \alpha(t-\eta))}{\partial x} f_1(\eta)\, \alpha\, d\eta + \int_0^t \frac{\partial M(x-1, \alpha(t-\eta))}{\partial x} f_2(\eta)\, \alpha\, d\eta, \tag{2.1}$$

where

$$M(\xi, \sigma) = \pi^{-1/2}\sigma^{-1/2} \sum_{n=-\infty}^{\infty} \exp\left\{-\frac{(\xi-2n)^2}{4\sigma}\right\}, \qquad \sigma > 0. \tag{2.2}$$

Applying Leibnitz's Rule for $0 < x < 1$, $0 < t \leqslant T$, we obtain

$$\frac{\partial u_\alpha}{\partial x}(0, t) = -\int_0^t \frac{\partial^2 M(x, \alpha(t-\eta))}{\partial x^2} f_1(\eta)\, \alpha\, d\eta + \int_0^t \frac{\partial^2 M(x-1, \alpha(t-\eta))}{\partial x^2} f_2(\eta)\, \alpha\, d\eta. \tag{2.3}$$

Since $f_1(0) = f_2(0) = 0$,

$$\frac{\partial M(x-\xi, \alpha(t-\eta))}{\partial \eta} = -\alpha \frac{\partial^2 M(x-\xi, \alpha(t-\eta))}{\partial x^2}, \qquad x \neq \xi, \quad t > \eta, \tag{2.4}$$

and

$$\lim_{\eta \uparrow t} M(x-\xi, \alpha(t-\eta)) = 0, \qquad x \neq \xi, \tag{2.5}$$

if f_1 and f_2 are continuously differentiable, then in (2.3) we can replace $\alpha(\partial^2 M/\partial x^2)$ by $-(\partial M/\partial\eta)$, integrate by parts and differentiate the result again with respect to x to obtain

$$\frac{\partial^2 u_\alpha}{\partial x^2} = -\int_0^t \frac{\partial M(x, \alpha(t-\eta))}{\partial x} f_1'(\eta)\, d\eta + \int_0^t \frac{\partial M(x-1, \alpha(t-\eta))}{\partial x} f_2'(\eta)\, d\eta. \tag{2.6}$$

Hence, we see that $\partial^2 u_\alpha/\partial x^2$ satisfies (1.4) with f_i, $i = 1, 2$, replaced respectively by $\alpha^{-1} f_i'$, $i = 1, 2$. From (2.6) we can obtain the following result.

Lemma 2.1. *If f_1 and f_2 are continuously differentiable with $f_1(0) = f_2(0) = 0$, $f_1' \geqslant 0, f_2' \geqslant 0$, and $f_1' > 0$ in each neighborhood of $t = 0$, then for $0 < x < 1$ and $0 < t \leqslant T$,*

$$\frac{\partial^2 u_\alpha}{\partial x^2}(x, t) > 0. \tag{2.7}$$

Proof. The result follows from (2.6) and the strong maximum principle for solutions of parabolic partial differential equations [2, 5, 6]. Q.E.D.

From [7] we see that

$$v_\alpha(x, t) = -\int_0^t M(x, \alpha(t-\eta))\, g_1(\eta)\, \alpha\, d\eta + \int_0^t M(x-1, \alpha(t-\eta))\, g_2(\eta)\, \alpha\, d\eta. \tag{2.8}$$

By an analysis similar to (2.1)–(2.6), we obtain

$$\frac{\partial^2 v_\alpha}{\partial x^2}(x, t) = -\int_0^t M(x, \alpha(t-\eta))\, g_1'(\eta)\, d\eta + \int_0^t M(x-1, \alpha(t-\eta))\, g_2'(\eta)\, d\eta \tag{2.9}$$

provided that $g_1(0) = g_2(0) = 0$ and that g_1 and g_2 are continuously differentiable.

LEMMA 2.2. *If g_1 and g_2 are continuously differentiable with $g_1(0) = g_2(0) = 0$, $g_1' \leqslant 0$, $g_2' \geqslant 0$, and $g_1' < 0$ in each neighborhood of $t = 0$, then for $0 < x < 1$, $0 < t \leqslant T$,*

$$\frac{\partial^2 v_\alpha}{\partial x^2}(x, t) > 0. \tag{2.10}$$

Proof. The result follows directly from $M > 0$ and (2.9). Q.E.D.

3. SOME COMPARISON RESULTS

We begin by defining for convenience

$$\beta(\tau) = 2\max\{\sup_{\substack{0\leqslant x\leqslant 1\\ 0\leqslant t\leqslant \tau}} |\, a(u(x, t)) - \alpha \,|,\ \sup_{\substack{0\leqslant x\leqslant 1\\ 0\leqslant t\leqslant \tau}} |\, a(v(x, t)) - \alpha \,|\}, \tag{3.1}$$

and

$$\alpha_\tau^{\pm} = \alpha \pm \beta(\tau). \tag{3.2}$$

Note that

$$\lim_{\tau\downarrow 0} \alpha_\tau^{\pm} = \alpha. \tag{3.3}$$

Consequently, for τ sufficiently small, say $0 < \tau \leqslant \tau_0$, $\alpha_\tau^{\pm} > 0$. We shall restrict τ so that $0 < \tau \leqslant \tau_0$. Consider functions $u_{\alpha_\tau^{\pm}}$ and $v_{\alpha_\tau^{\pm}}$ which arise as solutions of (1.4) and (1.5) via the replacement of α by $\alpha_\tau^{\pm}$. Also, we assume the hypotheses of Lemma 2.1 and Lemma 2.2.

LEMMA 3.1. *For τ sufficiently small and fixed, $0 \leqslant x \leqslant 1$ and $0 < t \leqslant \tau$,*

$$u_{\alpha_\tau^-}(x, t) \leqslant u(x, t) \leqslant u_{\alpha_\tau^+}(x, t), \tag{3.4}$$

and moreover, for $0 \leqslant t \leqslant \tau$,

$$\frac{\partial u_{\alpha_\tau^-}}{\partial x}(0, t) \leqslant \frac{\partial u}{\partial x}(0, t) \leqslant \frac{\partial u_{\alpha_\tau^+}}{\partial x}(0, t). \tag{3.5}$$

Proof. Consider first the case of $z^+ = u_{\alpha_\tau^+} - u$. Clearly, $z^+(0, t) = z^+(1, t) = z^+(x, 0) = 0$. But

$$\begin{aligned}\mathscr{L}z^+ \equiv a(u)\frac{\partial^2 z^+}{\partial x^2} - \frac{\partial z^+}{\partial t} &= a(u)\frac{\partial^2 u_{\alpha_\tau^+}}{\partial x^2} - \frac{\partial u_{\alpha_\tau^+}}{\partial t} \\ &= \{a(u) - \alpha_\tau^+\}\frac{\partial^2 u_{\alpha_\tau^+}}{\partial x^2} < 0\end{aligned} \tag{3.6}$$

since $\partial^2 u_{\alpha_\tau^+}/\partial x^2 > 0$ by Lemma 2.1 and $\{a(u) - \alpha_\tau^+\} < 0$ via (3.1) and (3.2). By the maximum principle [2, 5, 6], $z^+ \geqslant 0$. For the case of $z^- = u - u_{\alpha_\tau^-}$, we see that $z^-(0, t) = z^-(1, t) = z^-(x, 0) = 0$ and that

$$\mathscr{L}z^- = \{\alpha_\tau^- - a(u)\}\frac{\partial^2 u_{\alpha_\tau^-}}{\partial x^2} < 0 \tag{3.7}$$

since $\partial^2 u_{\alpha_\tau^-}/\partial x^2 > 0$ by Lemma 2.1 and $\{\alpha_\tau^- - a(u)\} < 0$ via (3.1) and (3.2). Hence, $z^- \geqslant 0$. Taken together, $z^\pm \geqslant 0$ is (3.4). The statement (3.5) follows from subtracting $f_1(t)$ from each side of (3.4), dividing by $x > 0$ and taking the limits as $x \to 0^+$, which all exist.

COROLLARY 3.1. *The inequality* (3.5) *holds for* $f_1' \geqslant 0$ *and* $f_2' \geqslant 0$.

Proof. As f_1 can be approximated uniformly by $\tilde{f}_1$ such that $\tilde{f}_1' \geqslant 0$ and $\tilde{f}_1' > 0$ in each neighborhood of $t = 0$, the result follows from the continuous dependence of u upon the data [1, 4]. Q.E.D.

LEMMA 3.2. *For* τ *sufficiently small and fixed,* $0 \leqslant x \leqslant 1$ *and* $0 < t \leqslant \tau$,

$$v_{\alpha_\tau^-}(x, t) \leqslant v(x, t) \leqslant v_{\alpha_\tau^+}(x, t) \tag{3.8}$$

provided that $g_1' < 0$ *and* $g_2' > 0$ *for* $0 < t \leqslant \tau$.

Proof. Let ϵ be a positive number satisfying $0 < \epsilon < 1$. Consider $v^\epsilon_{\alpha_\tau^-}$ which is obtained from (1.5) by substituting α_τ^- for α, $(1 - \epsilon)g_1$ for g_1 and $(1 - \epsilon)g_2$ for g_2. Set $z^\epsilon = v - v^\epsilon_{\alpha_\tau^-}$. Then, $z^\epsilon(x, 0) = 0$, $(\partial z^\epsilon/\partial x)(0, t) = \epsilon g_1(t) < 0$, and $(\partial z^\epsilon/\partial x)(1, t) = \epsilon g_2(t) > 0$. Hence, if z^ϵ had a negative minimum, then it would have to occur for $0 < x < 1$ and $0 < t \leqslant T$. But,

$$\mathscr{L}z^\epsilon = \{\alpha_\tau^- - a(v)\}\frac{\partial^2 v^\epsilon_{\alpha_\tau^-}}{\partial x^2} < 0, \tag{3.9}$$

since Lemma 2.2 implies that $\partial^2 v^\epsilon_{\alpha_\tau^-}/\partial x^2 > 0$ and $\{\alpha_\tau^- - a(v)\} < 0$ via (3.1) and (3.2), thus, $z^\epsilon \geqslant 0$ holds for each $\epsilon > 0$. The inequality $v_{\alpha_\tau^-} \leqslant v$ is obtained in the limit as ϵ tends to zero.

For the second half of (3.8), we consider $v^\epsilon_{\alpha_\tau^+}$ which is obtained from (1.5) by substituting α_τ^+ for α, $(1+\epsilon)g_1$ for g_1 and $(1+\epsilon)g_2$ for g_2. Set $z^\epsilon = v^\epsilon_{\alpha_\tau^+} - v$. Then, $z^\epsilon(x, 0) = 0$, $(\partial z^\epsilon/\partial x)(0, t) = \epsilon g_1(t) < 0$, and $(\partial z^\epsilon/\partial x)(1, t) = \epsilon g_2(t) > 0$. Hence, if z^ϵ had a negative minimum, it would have to occur in $0 < x < 1$, $0 < t \leqslant T$. But,

$$\mathscr{L} z^\epsilon = \{a(v) - \alpha_\tau^+\} \frac{\partial^2 v^\epsilon_{\alpha_\tau^+}}{\partial x^2} < 0 \tag{3.10}$$

since Lemma 2.2 implies that $\partial^2 v^\epsilon_{\alpha_\tau^+}/\partial x^2 > 0$ and $\{a(v) - a_\tau^+\} < 0$ via (3.1) and (3.2). Hence, $z^\epsilon \geqslant 0$ for each $\epsilon > 0$. The inequality $v \leqslant v_{\alpha_\tau^+}$ is obtained in the limit as ϵ tends to zero. Q.E.D.

COROLLARY 3.2. *The inequality* (3.8) *is valid if* $g_1' < 0$ *and* $g_2' > 0$ *is replaced by* $g_1' \leqslant 0$ *and* $g_2' \geqslant 0$.

Proof. As $g_1' \leqslant 0$ and $g_2' \geqslant 0$ can be approximated uniformly by functions $\tilde{g}_1'$ and $\tilde{g}_2'$ such that $\tilde{g}_1' < 0$ and $\tilde{g}_2' > 0$, the result follows from the continuous dependence of v upon the data [1, 4]. Q.E.D.

4. The Asymptotic Estimates

We begin our discussion by considering the inequality (3.5). To the hypotheses of Lemma 2.1 we add the assumption

$$f_1'(t) \geqslant f_2'(t), \qquad 0 < t \leqslant \tau. \tag{4.1}$$

This implies that the maximum of u_α for $0 \leqslant x \leqslant 1$, $0 \leqslant \eta \leqslant t$ occurs at $x = 0$ and $\eta = t$. Hence, the parabolic version of the Hopf Lemma [2, 5, 6] yields the fact that $(\partial u_\alpha/\partial x)(0, t) < 0$. Hence, dividing (3.5) through by $(\partial u_\alpha/\partial x)(0, t)$, we obtain

$$\frac{\partial u_{\alpha_\tau^-}}{\partial x}(0, t) \Big/ \frac{\partial u_\alpha}{\partial x}(0, t) \geqslant \left(\frac{\partial u}{\partial x}(0, t) \Big/ \frac{\partial u_\alpha}{\partial x}(0, t)\right) \geqslant \left(\frac{\partial u_{\alpha_\tau^+}}{\partial x}(0, t) \Big/ \frac{\partial u_\alpha}{\partial x}(0, t)\right). \tag{4.2}$$

Hence, we see that for $\tau > 0$, sufficiently small but fixed,

$$\begin{aligned} &\liminf_{t \to 0} \left\{\frac{\partial u_{\alpha_\tau^+}}{\partial x}(0, t) \Big/ \frac{\partial u_\alpha}{\partial x}(0, t)\right\} \\ &\quad \leqslant \liminf_{t \to 0} \left\{\frac{\partial u}{\partial x}(0, t) \Big/ \frac{\partial u_\alpha}{\partial x}(0, t)\right\} \\ &\quad \leqslant \limsup_{t \to 0} \left\{\frac{\partial u}{\partial x}(0, t) \Big/ \frac{\partial u_\alpha}{\partial x}(0, t)\right\} \leqslant \limsup_{t \to 0} \left\{\frac{\partial u_{\alpha_\tau^-}}{\partial x}(0, t) \Big/ \frac{\partial u_\alpha}{\partial x}(0, t)\right\}. \end{aligned} \tag{4.3}$$

We now derive a lower estimate on the first term in (4.3) and an upper estimate on the last term in (4.3). From (2.6) we see that

$$\frac{\partial u_\alpha}{\partial x}(0, t) = -\int_0^t M(0, \alpha(t-\eta)) f_1'(\eta)\, d\eta + \int_0^t M(-1, \alpha(t-\eta)) f_2'(\eta)\, d\eta \tag{4.4}$$

and

$$\frac{\partial u_{\alpha_\tau^+}}{\partial x}(0, t) = -\int_0^t M(0, \alpha_\tau^+(t-\eta)) f_1'(\eta)\, d\eta + \int_0^t M(-1, \alpha_\tau^+(t-\eta)) f_2'(\eta)\, d\eta. \tag{4.5}$$

First we see that

$$\begin{aligned}\left|\frac{\partial u_\alpha}{\partial x}(0, t)\right| &\leqslant \int_0^t M(0, \alpha(t-\eta)) f_1'(\eta)\, d\eta + \int_0^t M(-1, \alpha(t-\eta)) f_2'(\eta)\, d\eta \\ &= \int_0^t \frac{f_1'(\eta)}{\pi^{1/2}\alpha^{1/2}(t-\eta)^{1/2}} \left\{1 + 2\sum_{n=1}^{\infty} \exp\left\{\frac{-n^2}{\alpha(t-\eta)}\right\}\right. \\ &\quad \left. + \left(\frac{f_2'(\eta)}{f_1'(\eta)}\right) \sum_{n=-\infty}^{\infty} \exp\left\{\frac{-(2n-1)^2}{4\alpha(t-\eta)}\right\}\right\} d\eta. \end{aligned} \tag{4.6}$$

Set

$$\theta(\alpha, \tau) = 2\sum_{n=1}^{\infty} \exp\left\{\frac{-n^2}{\alpha\tau}\right\} + \sum_{n=-\infty}^{\infty} \exp\left\{\frac{-(2n-1)^2}{4\alpha\tau}\right\}. \tag{4.7}$$

Clearly,

$$\lim_{\tau\to 0} \theta(\alpha, \tau) = 0. \tag{4.8}$$

Thus,

$$\left|\frac{\partial u_\alpha}{\partial x}(0, t)\right| \leqslant (1 + \theta(\alpha, \tau)) \int_0^t \pi^{-1/2}\alpha^{-1/2}(t-\eta)^{-1/2} f_1'(\eta)\, d\eta \tag{4.9}$$

since $f_1'(\eta) > f_2'(\eta)$. For a lower bound on $|(\partial u_{\alpha_\tau^+}/\partial x)(0, t)|$ we see that for τ sufficiently small

$$\begin{aligned}\left|\frac{\partial u_{\alpha_\tau^+}}{\partial x}(0, t)\right| &= \int_0^t \frac{f_1'(\eta)}{\pi^{1/2}\alpha_\tau^{+1/2}(t-\eta)^{1/2}} \left\{1 + 2\sum_{n=1}^{\infty} \exp\left\{\frac{-n^2}{\alpha_\tau^+(t-\eta)}\right\}\right. \\ &\quad \left. - \left(\frac{f_2'(\eta)}{f_1'(\eta)}\right) \sum_{n=-\infty}^{\infty} \exp\left\{\frac{-(2n-1)^2}{4\alpha_\tau^+(t-\eta)}\right\}\right\} d\eta \\ &\geqslant \{1 - \theta(\alpha_\tau^+, \tau)\} \int_0^t \pi^{-1/2}(\alpha_\tau^+)^{-1/2}(t-\eta)^{-1/2} f_1'(\eta)\, d\eta > 0. \end{aligned} \tag{4.10}$$

For $0 < t \leqslant \tau$, we see that

$$\left(\frac{\partial u_{\alpha_\tau^+}(0, t)}{\partial x} \Big/ \frac{\partial u_\alpha(0, t)}{\partial x}\right) \geqslant \frac{\{1 - \theta(\alpha_\tau^+, \tau)\}\, \alpha^{1/2}}{\{1 + \theta(\alpha, \tau)\}\, (\alpha_\tau^+)^{1/2}}. \tag{4.11}$$

In a similar manner, it follows that for $0 < t \leqslant \tau$,

$$\left(\frac{\partial u_{\alpha_\tau^-}(0, t)}{\partial x} \Big/ \frac{\partial u_\alpha}{\partial x}(0, t)\right) \leqslant \frac{\{1 + \theta(\alpha_\tau^-, \tau)\}\, \alpha^{1/2}}{\{1 - \theta(\alpha, \tau)\}\, (\alpha_\tau^-)^{1/2}}. \tag{4.12}$$

Thus, from (4.3)

$$\begin{aligned} \frac{\{1 - \theta(\alpha_\tau^+, \tau)\}\, \alpha^{1/2}}{\{1 + \theta(\alpha, \tau)\}\, \alpha_\tau^{+1/2}} &\leqslant \liminf_{t\to 0} \left(\frac{\partial u}{\partial x}(0, t) \Big/ \frac{\partial u_\alpha}{\partial x}(0, t)\right) \\ &\leqslant \limsup_{t\to 0} \left(\frac{\partial u}{\partial x}(0, t) \Big/ \frac{\partial u_\alpha}{\partial x}(0, t)\right) \leqslant \frac{\{1 + \theta(\alpha_\tau^-, \tau)\}\, \alpha^{1/2}}{\{1 - \theta(\alpha, \tau)\}\, \alpha_\tau^{-1/2}} \end{aligned} \tag{4.13}$$

holds for each τ in $0 < \tau \leqslant \tau_0$ where τ_0 is sufficiently small but fixed. Taking the limit as τ tends to zero, it follows from (3.3) and (4.8) that

$$\begin{aligned} 1 &\leqslant \liminf_{t\to 0} \left(\frac{\partial u}{\partial x}(0, t) \Big/ \frac{\partial u_\alpha}{\partial x}(0, t)\right) \\ &\leqslant \limsup_{t\to 0} \left(\frac{\partial u}{\partial x}(0, t) \Big/ \frac{\partial u_\alpha}{\partial x}(0, t)\right) \leqslant 1. \end{aligned} \tag{4.14}$$

We can summarize the result in the following statement.

THEOREM 1. *If f_1 and f_2 are continuously differentiable functions with $f_1' > f_2' \geqslant 0$ on $0 \leqslant t \leqslant T$ and with $f_1(0) = f_2(0) = 0$, then*

$$\lim_{t\to 0} \left(\frac{\partial u}{\partial x}(0, t) \Big/ \frac{\partial u_\alpha}{\partial x}(0, t)\right) = 1, \tag{4.15}$$

where u is the solution of (1.1), *u_α is the solution of* (1.6) *and α is defined by* (1.5).

Under the assumptions of Lemma 2.2, it follows from Corollary 3.2 that (3.8) is valid. Moreover, from (2.8), it follows that $v_\alpha > 0$. Hence, for $x = 0$, we can divide (3.8) by $v_\alpha(0, t)$ to obtain

$$\frac{v_{\alpha_\tau^-}(0, t)}{v_\alpha(0, t)} \leqslant \frac{v(0, t)}{v_\alpha(0, t)} \leqslant \frac{v_{\alpha_\tau^+}(0, t)}{v_\alpha(0, t)}. \tag{4.16}$$

As above, we have for $\tau > 0$ and sufficiently small

$$\begin{aligned} \liminf_{t\to 0} \frac{v_{\alpha_\tau^-}(0, t)}{v_\alpha(0, t)} &\leqslant \liminf_{t\to 0} \frac{v(0, t)}{v_\alpha(0, t)} \\ &\leqslant \limsup_{t\to 0} \frac{v(0, t)}{v_\alpha(0, t)} \leqslant \limsup_{t\to 0} \frac{v_{\alpha_\tau^+}(0, t)}{v_\alpha(0, t)}. \end{aligned} \tag{4.17}$$

Under the additional assumption that $-g_1' > g_2'$ and with an analysis similar to that above we obtain the estimates

$$\liminf_{t\to 0} \frac{v_{\alpha_\tau^-}(0, t)}{v_\alpha(0, t)} \geqslant \frac{\{1 - \theta(\alpha_\tau^-, \tau)\}\, \alpha^{1/2}}{\{1 + \theta(\alpha, \tau)\}\, (\alpha_\tau^-)^{1/2}} \tag{4.18}$$

and

$$\limsup_{t\to 0} \frac{v_{\alpha_\tau}(0, t)}{v_\alpha(0, t)} \leqslant \frac{\{1 + \theta(\alpha_\tau^+, \tau)\}\, \alpha^{1/2}}{\{1 - \theta(\alpha, \tau)\}\, (\alpha_\tau^+)^{1/2}}\,. \tag{4.19}$$

Again, it follows from (3.3) and (4.8) that as τ tends to zero,

$$1 \leqslant \liminf_{t\to 0} \frac{v(0, t)}{v_\alpha(0, t)} \leqslant \limsup_{t\to 0} \frac{v(0, t)}{v_\alpha(0, t)} \leqslant 1. \tag{4.20}$$

Consequently, we can summarize the result in the following statement.

THEOREM 2. *If* g_1 *and* g_2 *are continuously differentiable functions with* $-g_1' > g_2' \geqslant 0$ on $0 \leqslant t \leqslant T$ and with $g_1(0) = g_2(0) = 0$, then

$$\lim_{t\to 0} \frac{v(0, t)}{v_\alpha(0, t)} = 1, \tag{4.21}$$

where v *is the solution of* (1.2), v_α *is the solution of* (1.7) *and* α *is defined by* (1.5).

We turn now to the asymptotic estimates involving U_α and V_α. It is easy to see that

$$\frac{\partial U_\alpha}{\partial x}(0, t) = -\int_0^t \frac{f_1'(\eta)\, d\eta}{\pi^{1/2}\alpha^{1/2}(t - \eta)^{1/2}}\,. \tag{4.22}$$

Hence, from (4.4), (4.6), (4.7), we see that for $\tau > 0$, sufficiently small and fixed,

$$1 - \theta(\alpha, \tau) \leqslant \left\{\frac{\partial u_\alpha}{\partial x}(0, t) \Big/ \frac{\partial U_\alpha}{\partial x}(0, t)\right\} \leqslant 1 + \theta(\alpha, \tau) \tag{4.22}$$

holds for all $0 < t \leqslant \tau$. Consequently,

$$\lim_{t\downarrow 0} \left\{\frac{\partial u_\alpha}{\partial x}(0, t) \Big/ \frac{\partial U_\alpha}{\partial x}(0, t)\right\} = 1. \tag{4.23}$$

A similar argument yields

$$\lim_{t\downarrow 0} \frac{v_\alpha(0, t)}{V_\alpha(0, t)} = 1. \tag{4.24}$$

Thus, we can summarize the results in the following statements.

THEOREM 3. *If f_1 and f_2 are continuously differentiable functions with $f_1' > f_2' \geqslant 0$ on $0 \leqslant t \leqslant T$ and with $f_1(0) = f_2(0) = 0$, then*

$$\lim_{t \downarrow 0} \left\{ \frac{\partial u}{\partial x}(0, t) \Big/ \frac{\partial U_\alpha}{\partial x}(0, t) \right\} = 1, \tag{4.25}$$

where u is the solution of (1.1), U_α is the solution of (1.8) and α is defined by (1.5).

Proof. The result follows immediately from (4.15) and (4.23).

THEOREM 4. *If g_1 and g_2 are continuously differentiable functions with $-g_1' > g_2' \geqslant 0$ on $0 \leqslant t \leqslant T$ and with $f_1(0) = g_2(0)$, then*

$$\lim_{t \downarrow 0} \frac{v(0, t)}{V_\alpha(0, t)} = 1, \tag{4.26}$$

where v is the solution of (1.2), V_α is the solution of (1.9) and α is defined by (1.5).

Proof. The result follows immediately from (4.21) and (4.24).

We turn our attention now to the asymptotic behavior of $(\partial U/\partial x)(0, t)$ and $V(0, t)$. Consider $U_n = U_n(x, t)$ and $V_n = V_n(x, t)$ which solve, respectively,

$$\begin{aligned}
&\frac{\partial U_n}{\partial t} = a(U) \frac{\partial^2 U_n}{\partial x^2}, && 0 < x < n, \quad 0 < t \leqslant T, \\
&U_n(x, 0) = 0, && 0 \leqslant x \leqslant n, \\
&U_n(0, t) = f_1(t), && 0 < t \leqslant T, \\
&U_n(n, t) = 0, && 0 < t \leqslant T,
\end{aligned} \tag{4.27}$$

and

$$\begin{aligned}
&\frac{\partial V_n}{\partial t} = a(V) \frac{\partial^2 V_n}{\partial x^2}, && 0 < x < n, \quad 0 < t - T, \\
&V_n(x, 0) = 0, && 0 \leqslant x \leqslant n, \\
&\frac{\partial V_n}{\partial x}(0, t) = g_1(t), && 0 < t \leqslant T, \\
&V_n(n, t) = 0, && 0 < t \leqslant T.
\end{aligned} \tag{4.28}$$

An elementary application of the maximum principle for parabolic partial differential equations shows that for each (x, t) the sequences $\{U_n(x, t)\}$ and $\{V_n(x, t)\}$ are monotone increasing and are bounded above uniformly in n, x and t. Application of the interior Schauder estimates [1, 4] and a compactness argument based on the Ascoli-Arzela theorem [3] yields that

$$U(x, t) = \lim_{n \to \infty} U_n(x, t) \qquad \text{and} \qquad V(x, t) = \lim_{n \to \infty} V_n(x, t).$$

Since we are interested in the behavior of U and V when they are nearly zero, it is no loss of generality to assume that $a(\cdot)$ is bounded above. A repetition of the arguments of Sections 2 and 3 yields that U_n and V_n are bounded above by solutions of the heat equation with a diffusivity exceeding the bound for $a(\cdot)$ and boundary and initial conditions as in (4.27) and (4.28). Taking limits as n tends to infinity, we obtain that U and V are bounded above by U_α and V_α, where α here is a constant exceeding the bound for $a(\cdot)$. The sole purpose of the preceeding remarks has been to demonstrate that as t tends to zero U and V tend uniformly to zero. Hence, the analysis of Section 3 can be applied uniformly to U_n and V_n to yield comparison results analogous to Lemma 3.1 and Lemma 3.2. The analysis of the first part of this section simplifies due to the simpler formulas for the heat equation in a quarter plane. The results can be expressed as follows.

THEOREM 5. *If f_1 is continuously differentiable with $f_1' > 0$ and $0 < t \leqslant t$ with $f_1(0) = 0$, then*

$$\lim_{t \downarrow 0} \left\{ \frac{\partial U}{\partial x}(0, t) \Big/ \frac{\partial U_\alpha}{\partial x}(0, t) \right\} = 1, \tag{4.29}$$

where U is the solution of (1.3), *U_α is the solution of* (1.8) *and α is defined by* (1.5).

THEOREM 6. *If g_1 is continuously differentiable with $-g' > 0$ on $0 < t \leqslant T$ and with $g_1(0) = 0$, then*

$$\lim_{t \downarrow 0} \frac{V(0, t)}{V_\alpha(0, t)} = 1, \tag{4.30}$$

where V is the solution of (1.4), *V_α is the solution of* (1.9) *and α is defined by* (1.5).

Remark. Clearly, the Theorems 1 to 6 can be compiled into the equations (1.10) and (1.11).

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